Teoria

Twierdzenie 1 (Reguła de l'Hospitala) Niech f i g będą funkcjami różniczkowalnymi w pewnym sąsiedztwie U punktu x_0 oraz $g'(x) \neq 0$ dla $x \in U$. Jeżeli
$\lim\limits_{x\to x_0} f(x) = \lim\limits_{x\to x_0} g(x) = 0$ albo $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \pm\infty$
oraz istnieje granica $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)}$ (właściwa lub nie), to istnieje rownież granica $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$, przy czym

$$\left\{\frac{-\infty/+\infty}{-\infty/+\infty}\right\} \left\{\frac{0}{0}\right\} = \lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}.$$

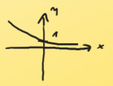
Uwaga Twierdzenie 1 zachodzi także dla granic jednostronnych w punkcie.

Twierdzenie 2 Regułę de l'Hospitala można stosować również do obliczania granic funkcji w przypadku symboli nieoznaczonych innych niż $\frac{0}{0}$ i $\frac{\infty}{\infty}$; dokonując najpierw podanych niżej przekształceń.

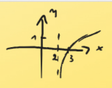
Symbol przed przekształceniem	Przekształcenie	Symbol po przekształceniu
$\infty - \infty$	$f - g = \begin{cases} f(1-\frac{g}{f}) \\ \frac{\frac{1}{g}-\frac{1}{f}}{\frac{1}{fg}} \end{cases}$	$\infty(1-\frac{\infty}{\infty})$ $\frac{0}{0}$
$0 \cdot \infty$	$f \cdot g = \begin{cases} \frac{f}{\frac{1}{g}} \\ \frac{g}{\frac{1}{f}} \end{cases}$	$\frac{0}{0}$ $\frac{\infty}{\infty}$
$0^0,\ \infty^0,\ 1^\infty$	$f^g = e^{g \ln f}$	$e^0 \cdot (\pm\infty)$

$y = \sin x$

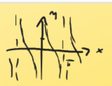
$y = e^x$

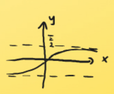

$y = e^{-x}$

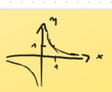
$y = \ln x$


$y = \ln(x-2)$

$y = \cos x$


$y = \operatorname{ctg} x$


$y = \operatorname{arctg} x$


$y = \frac{1}{x}$

$y =$

Przykład

Przykład 1 Obliczymy granicę $\lim\limits_{x\to 0} \frac{\sin(4x)}{7x}$.

$\cos(4x) \cdot (4x)'$

$$\lim_{x\to 0} \frac{\sin(4x)}{7x} = \left\{\frac{\sin(0)}{0}\right\} = \left\{\frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0} \frac{(\sin(4x))'}{(7x)'} = \lim_{x\to 0} \frac{4\cos(4x)}{7} = = \left\{\frac{4\cos(0)}{7} = \frac{4}{7}\right\} = \frac{4}{7}$$

Zadania

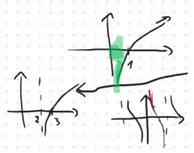
Zadanie 1 Obliczymy granice:

$$\lim_{x\to+\infty} \frac{3x^2-4x+6}{x^2+5} = \left\{\frac{\infty - \infty + 6}{\infty + 5}\right\} = \lim_{x\to\infty} \frac{x^2\left(3 - \frac{4}{x} + \frac{6}{x^2}\right)}{x^2+5} = \left\{\frac{\infty \cdot 3}{\infty} = \frac{\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to+\infty} \frac{6x-4}{2x} = \left\{\frac{\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to+\infty} \frac{6}{2} = \frac{6}{2}$$

$$\lim_{x\to-\infty} \frac{e^x - e^{-x}}{3x} = \left\{\frac{e^{-\infty} - e^{-(-\infty)}}{-\infty} = \frac{0 - e^{+\infty}}{-\infty} = \frac{-\infty}{-\infty}\right\} \overset{H}{=} \lim_{x\to-\infty} \frac{(e^x - e^{-x})'}{(3x)'} = \lim_{x\to-\infty} \frac{e^x + e^{-x}}{3} = \left\{\frac{0+\infty}{3}\right\} = \infty$$

$$\lim_{x\to 0} \frac{e^x - e^{-x}}{4x} = \left\{\frac{e^0 - e^0}{0} = \frac{1-1}{0} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0} \frac{e^x + e^{-x}}{4} = \left\{\frac{e^0 + e^0}{4} = \frac{2}{4}\right\} = \frac{1}{2}$$

$$\lim_{x\to+\infty} \frac{2\ln(x)+1}{x^2} = \left\{\frac{\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to\infty} \frac{(2\ln x + 1)'}{(x^2)'} = \lim_{x\to\infty} \frac{\frac{2}{x}}{2x} = \lim_{x\to\infty} \frac{2}{2x^2} = \left\{\frac{2}{\infty}\right\} = 0$$



$$\lim_{x\to 3} \frac{\ln(x-2)}{x^2-x-6} = \left\{\frac{\ln(3-2)}{9-3-6} = \frac{\ln 1}{0} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 3} \frac{\frac{1}{x-2}\cdot 1}{2x-1} = \left\{\frac{\frac{1}{3-2}}{6-1} = \frac{1}{5}\right\} = \frac{1}{5}$$

$$\lim_{x\to 0^+} \frac{\ln(x)}{\operatorname{ctg}(x)} = \left\{\frac{\ln 0^+}{\operatorname{ctg} 0^+} = \frac{-\infty}{+\infty}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}} = \lim_{x\to 0^+} \frac{\sin^2 x}{-x} = \left\{\frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{2\sin x \cdot \cos x}{-1} = \left\{\frac{2\cdot 0\cdot 1}{-1}\right\} = 0$$

$$\lim_{x\to 0} \frac{\cos(x)-1}{\sin(x)} = \left\{\frac{\cos 0 - 1}{\sin 0} = \frac{1-1}{0} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0} \frac{-\sin x}{\cos x} = \left\{\frac{-0}{1}\right\} = 0$$

Zadanie 2 Obliczymy granice:

$\sin^2 x = (\sin x)^2$

$$\lim_{x\to-\infty} xe^x = \left\{-\infty \cdot e^{-\infty} = -\infty \cdot 0\right\} = \lim_{x\to-\infty} \frac{x}{e^{-x}} = \left\{\frac{-\infty}{e^{+\infty}} = \frac{-\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to-\infty} \frac{1}{-e^{-x}} = \left\{\frac{1}{-e^{+\infty}} = \frac{1}{-\infty}\right\} = 0$$

Uwaga: Zapisujemy iloczyn jako iloraz: $xe^x = \frac{x}{e^{-x}}$

$\frac{1}{e^{-x}} = e^x$

$$\lim_{x\to 0^+} x^3 \ln(x) = \left\{0 \cdot \ln 0^+ = 0 \cdot (-\infty)\right\} = \lim_{x\to 0^+} \frac{\ln x}{x^{-3}} = \left\{\frac{\ln 0^+}{(\frac{1}{0^+})^3} = \frac{-\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-3x^{-4}} = \lim_{x\to 0^+} \frac{1 \cdot x^4}{x(-3)} = \lim_{x\to 0^+} \frac{x^3}{-3} = 0$$

$\frac{1}{x^4}$

Uwaga: Zapisujemy iloczyn jako iloraz: $x^3 \ln(x) = \frac{\ln(x)}{x^{-3}}$

$$\lim_{x\to 0^+} \sin(x)\ln(x)$$

$$= \left\{0 \cdot (-\infty)\right\} = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{\sin x}} = \left\{\frac{-\infty}{\frac{1}{0^+}} = \frac{-\infty}{+\infty}\right\} =$$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{\sin^2 x}\cdot\cos x} = \lim_{x\to 0^+} \frac{-\sin^2 x}{x\cos x} = \left\{\frac{0}{0}\right\} \overset{H}{=}$$

$$= \lim_{x\to 0^+} \frac{-2\sin x\cos x}{\cos x + x(-\sin x)} = \left\{\frac{0}{1-0}\right\} = 0$$

$$\lim_{x\to+\infty} \frac{2\ln(x)+1}{x^2}$$

$$\lim_{x\to+\infty} (x^2 - \ln(x)) = \left\{\infty - \infty\right\} = \lim_{x\to\infty} x^2\left(1 - \boxed{\frac{\ln x}{x^2}}\right) = \left\{\infty\left(1 - \boxed{\frac{\infty}{\infty}}\right)\right\} = \left\{\infty(1 - 0) = \infty \cdot 1\right\} = \infty$$

$$\lim_{x\to 0^+} \left(1 + xe^{\frac{1}{\sqrt{x}}}\right)$$

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \left\{\frac{\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{x\cdot 2x} = \left\{\frac{1}{\infty}\right\} = 0$$

Uwaga: Zapisujemy różnicę jako iloczyn wyłączając x^2 przed nawias: $(x^2 - \ln(x)) = x^2\left(1 - \frac{\ln(x)}{x^2}\right)$

Zadanie 3 Obliczymy granice:

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)}\right)$$

$$\lim_{x\to 1} \left(\frac{1}{(x-1)^2} - \frac{1}{\ln(x)}\right)$$

$(\operatorname{arctg} x)' = \frac{1}{1+x^2}$

$$\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\operatorname{arctg} x}\right) = \left\{\frac{1}{0} - \frac{1}{0}\right\} = \lim_{x\to 0} \frac{\operatorname{arctg} x - x}{x \cdot \operatorname{arctg} x} = \left\{\frac{0-0}{0\cdot 0} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0} \frac{\frac{1}{1+x^2} - 1}{1\cdot\operatorname{arctg} x + x\cdot\frac{1}{1+x^2}} = \left\{\frac{1-1}{0+0} = \frac{0}{0}\right\} = \lim_{x\to 0} \frac{\frac{1-(1+x^2)}{\cancel{1+x^2}}}{\frac{(1+x^2)\operatorname{arctg} x + x}{\cancel{1+x^2}}} = \lim_{x\to 0} \frac{-x^2}{(1+x^2)\operatorname{arctg} x + x} = \left\{\frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0} \frac{-2x}{2x\operatorname{arctg} x + (1+x^2)\cdot\frac{1}{\cancel{1+x^2}} + 1} = \lim_{x\to 0} \frac{-2x}{2x\operatorname{arctg} x + 2} = \left\{\frac{0}{0+2} = \frac{0}{2}\right\} = 0$$

Zadanie 4 Obliczymy granice:

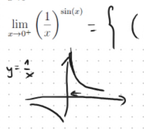
$$\lim_{x\to 0^+} x^{2x}$$

Uwaga: Wykorzystujemy własności funkcji wykładniczych i logarytmicznych i zapisujemy: $x^{2x} = e^{\ln(x^{2x})}$

$$\lim_{x\to 0^+} (\operatorname{arctg}(x))^x$$

$$\lim_{x\to 0^+} \left(\frac{1}{x}\right)^{\sin(x)} = \left\{\left(\frac{1}{0^+}\right)^{\sin 0^+} = (+\infty)^0\right\} = \lim_{x\to 0^+} e^{\ln\left(\frac{1}{x}\right)^{\sin x}} = \lim_{x\to 0^+} e^{\boxed{\sin x \cdot \ln\frac{1}{x}}} = \left\{e^{\sin 0 \cdot \ln\frac{1}{0^+}} = e^{0\cdot\ln\infty} = e^{\boxed{0\cdot\infty}}\right\} = \left\{e^0\right\} = 1$$

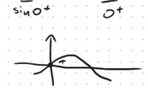
$y = \frac{1}{x}$



$[f(x)]^{g(x)}$ D: $f(x) > 0$, $g(x) > 0$

$$\lim_{x\to 0^+} \sin x \cdot \ln\frac{1}{x} = \lim_{x\to 0^+} \frac{\sin x}{\frac{1}{\ln\frac{1}{x}}} = \left\{\frac{0}{\frac{1}{\infty}} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{\cos x}{-\frac{1}{(\ln\frac{1}{x})^2}\cdot(\ln\frac{1}{x})'} = \lim_{x\to 0^+} \frac{\cos x}{-\frac{1}{(\ln(\frac{1}{x}))^2}\cdot\frac{1}{\frac{1}{x}}\cdot(\frac{1}{x})'}$$

$$\lim_{x\to 0^+} \sin x \ln\frac{1}{x} = \lim_{x\to 0^+} \frac{\ln\frac{1}{x}}{\frac{1}{\sin x}} = \left\{\frac{\ln\frac{1}{0^+}}{\frac{1}{\sin 0^+}} = \frac{\ln\infty}{\frac{1}{0^+}} = \frac{\infty}{\infty}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{\frac{1}{\frac{1}{x}}\cdot\left(-\frac{1}{x^2}\right)}{-\frac{1}{(\sin x)^2}\cdot\cos x} = \lim_{x\to 0^+} \frac{\sin^2 x}{x\cos x} = \left\{\frac{0}{0\cdot 1} = \frac{0}{0}\right\} \overset{H}{=} \lim_{x\to 0^+} \frac{2\sin x\cos x}{1\cdot\cos x + x\cdot(-\sin x)} = \left\{\frac{2\cdot 0\cdot 1}{1\cdot 1 + 0\cdot(-0)} = \frac{0}{1}\right\} = 0$$

$= -\frac{x}{x^2} = -\frac{1}{x}$

$(\sin x)'$