30.10.2025r.

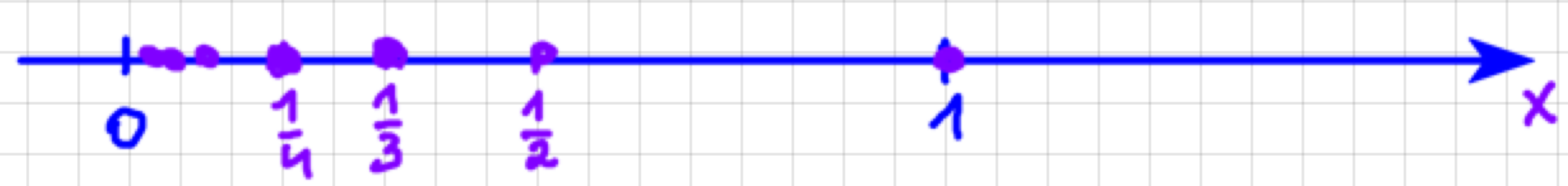
Granice ciągów liczbowych.

$a_1, a_2, a_3, a_4, \ldots, a_n \qquad n \in \mathbb{N} = \{1, 2, 3, \ldots\}$

$$\lim_{n \to \infty} a_n$$

Rozważmy ciąg o wyrazie ogólnym $a_n = \frac{1}{n}$.

$a_1 = \frac{1}{1} = 1, \quad a_2 = \frac{1}{2}, \quad a_3 = \frac{1}{3}, \quad a_4 = \frac{1}{4}, \quad \ldots$



$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n} = \left[\frac{1}{\infty}\right] = 0 \qquad a_n \xrightarrow[n \to \infty]{} 0$$

$$\lim_{n\to\infty} \frac{1}{n^2} = \left[\frac{1}{\infty^2}\right] = \left[\frac{1}{\infty}\right] = 0$$

$$\lim_{n\to\infty} \frac{1}{n^3} = \left[\frac{1}{\infty^3}\right] = \left[\frac{1}{\infty}\right] = 0$$

a^b

np. $a_n = n^5$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^5 = [\infty^5] = \infty$$

Przykład Oblicz granice ciągów:

a) $a_n = n^3 + 2n^2$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (n^3 + 2n^2) = [\infty^3 + 2\cdot\infty^2] = [\infty + \infty] = \infty$$

b) $a_n = 5n^2 - 2n$

symbol nieoznaczony

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (5n^2 - 2n) = [5\cdot\infty^2 - 2\cdot\infty] = [\infty - \infty] = \lim_{n\to\infty} \underbrace{n^2}_{\to\infty} \cdot \Big(5 - \underbrace{\frac{2}{n}}_{\to 0}\Big) =$$

$$= [\infty \cdot 5] = \infty$$

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c) $$a_n = \frac{3n+1}{5n^2+2}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n+1}{5n^2+2} = \left[\frac{\infty}{\infty}\right] = \lim_{n\to\infty} \frac{\frac{3n}{n^2}+\frac{1}{n^2}}{\frac{5n^2}{n^2}+\frac{2}{n^2}} = \lim_{n\to\infty} \frac{\frac{3}{n}+\frac{1}{n^2}}{5+\frac{2}{n^2}} = \frac{0}{5} = 0$$

symbol nieoznaczony

$\frac{3}{n} \to 0$, $\frac{1}{n^2} \to 0$, $\frac{2}{n^2} \to 0$

d) $$a_n = \frac{5n^3-7n^2}{6n^3+8}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{5n^3-7n^2}{6n^3+8} = \left[\frac{\infty-\infty}{\infty}\right] = \lim_{n\to\infty} \frac{\frac{5n^3}{n^3}-\frac{7n^2}{n^3}}{\frac{6n^3}{n^3}+\frac{8}{n^3}} = \lim_{n\to\infty} \frac{5-\frac{7}{n}}{6+\frac{8}{n^3}} = \frac{5}{6}$$

$\frac{7}{n} \to 0$, $\frac{8}{n^3} \to 0$

e) $$a_n = \frac{2n^2+n}{3-n}$$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n^2+n}{3-n} = \left[\frac{\infty}{-\infty}\right] = \lim_{n\to\infty} \frac{\frac{2n^2}{n}+\frac{n}{n}}{\frac{3}{n}-\frac{n}{n}} = \lim_{n\to\infty} \frac{2n+1}{\frac{3}{n}-1} =$$

$$= \left[\frac{\infty}{-1}\right] = -\infty$$

$2n \to \infty$, $\frac{3}{n} \to 0$

f) $a_n = \dfrac{2^{n+1}+3^{2n-1}}{4^n+9^n}$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2^{n+1}+3^{2n-1}}{4^n+9^n} = \left[\frac{2^\infty+3^\infty}{4^\infty+9^\infty}\right] = \left[\frac{\infty+\infty}{\infty+\infty}\right] = \left[\frac{\infty}{\infty}\right] =$$

$$= \lim_{n\to\infty} \frac{2^n \cdot 2^1 + (3^2)^n \cdot 3^{-1}}{4^n+9^n} = \lim_{n\to\infty} \frac{2^n \cdot 2 + 9^n \cdot 3^{-1}}{4^n+9^n} = \lim_{n\to\infty} \frac{\frac{2^n \cdot 2}{9^n} + \frac{\cancel{9^n} \cdot 3^{-1}}{\cancel{9^n}}}{\frac{4^n}{9^n} + \frac{9^n}{9^n}} =$$

$$= \lim_{n\to\infty} \frac{\overbrace{\left(\frac{2}{9}\right)^n}^{\to 0} \cdot 2 + 3^{-1}}{\left(\frac{4}{9}\right)^n + 1} = \frac{3^{-1}}{1} = \frac{1}{3}$$

$$a^n \cdot b^n = (a \cdot b)^n$$

g) $$\lim_{n\to\infty} \frac{6^n + 2^n \cdot 3^{n-1} + 5^n}{4^n - 2\cdot 6^{n-1}} = \lim_{n\to\infty} \frac{6^n + 2^n \cdot 3^n \cdot 3^{-1} + 5^n}{4^n - 2\cdot 6^n \cdot 6^{-1}} =$$

$$= \lim_{n\to\infty} \frac{6^n + (2\cdot 3)^n \cdot 3^{-1} + 5^n}{4^n - 2\cdot 6^n \cdot 6^{-1}} = \lim_{n\to\infty} \frac{6^n + 6^n \cdot 3^{-1} + 5^n}{4^n - 2\cdot 6^n \cdot 6^{-1}} =$$

$$= \lim_{n\to\infty} \frac{\frac{6^n}{6^n} + \frac{\cancel{6^n}^{\,1} \cdot 3^{-1}}{\cancel{6^n}_{\,1}} + \frac{5^n}{6^n}}{\frac{4^n}{6^n} - \frac{2\cdot \cancel{6^n}^{\,1} \cdot 6^{-1}}{\cancel{6^n}_{\,1}}} = \lim_{n\to\infty} \frac{1 + 3^{-1} + \underbrace{\left(\frac{5}{6}\right)^n}_{\to 0}}{\underbrace{\left(\frac{4}{6}\right)^n}_{\to 0} - 2\cdot 6^{-1}} = \frac{1 + \frac{1}{3}}{-2\cdot\frac{1}{6}} =$$

$$= \frac{\frac{4}{\cancel{3}_{\,1}}}{-\frac{1}{\cancel{3}_{\,1}}} = -\frac{4}{1} = -4$$

h) $$\lim_{n\to\infty} \sqrt[n]{5^n} = \lim_{n\to\infty} \left(5^n\right)^{\frac{1}{n}} = \lim_{n\to\infty} 5^{\cancel{n}\cdot\frac{1}{\cancel{n}}} = 5.$$

i) $$\lim_{n\to\infty} \sqrt[n]{2^n+3^n+4^n} = \left| \begin{array}{l} \text{Tw.} \\ \bullet\ \lim\limits_{n\to\infty} \sqrt[n]{a} \overset{a>0}{=} 1 \\ \bullet\ \lim\limits_{n\to\infty} \sqrt[n]{n} = 1 \end{array} \right| =$$

$$= \lim_{n\to\infty} \sqrt[n]{4^n\cdot\left(\frac{2^n}{4^n}+\frac{3^n}{4^n}+1\right)} = \lim_{n\to\infty} \underbrace{\sqrt[n]{4^n}}_{4} \cdot \overbrace{\sqrt[n]{\underbrace{\left(\tfrac{2}{4}\right)^n}_{0}+\underbrace{\left(\tfrac{3}{4}\right)^n}_{0}+1}}^{\to 1} =$$

$$= 4\cdot 1 = 4$$

j) $$\lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right) = [\infty-\infty] = \lim_{n\to\infty}\left(\sqrt{n}-\sqrt{n+1}\right)\cdot\frac{\left(\sqrt{n}+\sqrt{n+1}\right)}{\left(\sqrt{n}+\sqrt{n+1}\right)} =$$

$$= \lim_{n\to\infty} \frac{\left(\sqrt{n}-\sqrt{n+1}\right)\cdot\left(\sqrt{n}+\sqrt{n+1}\right)}{\sqrt{n}+\sqrt{n+1}} = \lim_{n\to\infty}\frac{\cancel{n}-(\cancel{n}+1)}{\sqrt{n}+\sqrt{n+1}} = \lim_{n\to\infty}\frac{-1}{\sqrt{n}+\sqrt{n+1}} = \left[\frac{-1}{\infty}\right] = 0$$