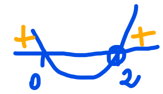
D Asymptoty pionowe Asymptoty poziome Asymptoty ukośne

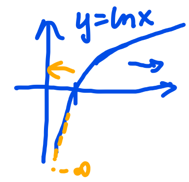
7 Asymptoty (cz.2)

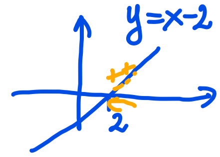
Zadanie 1. Wyznaczyć asymptoty wykresu funkcji $f(x) = \ln \frac{x}{x-2}$

$D: \ x \neq 2 \wedge \frac{x}{x-2} > 0$ $\qquad \frac{x}{x-2} > 0 \Rightarrow x(x-2) > 0$



$D = (-\infty, 0) \cup (2, +\infty)$





• As. pionowe

$\lim_{x \to 0^-} f(x) = \left[\ln \frac{0^-}{-2}\right] = [\ln 0^+] = -\infty$

$\lim_{x \to 2^+} f(x) = \left[\ln \frac{2}{0^+}\right] = [\ln(+\infty)] = +\infty$

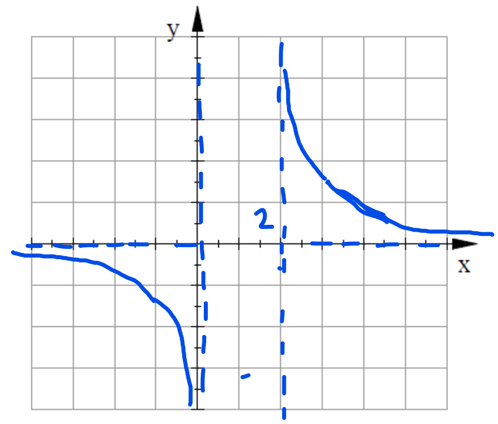
$x=0$ - as. pionowa lewostronna, $x=2$ - as. pionowa prawostronna

• As. poziome

$\lim_{x\to+\infty} f(x) = \left\{ \lim_{x\to+\infty} \frac{x}{x-2} = \lim_{x\to+\infty} \frac{\cancel{x}\,1}{\cancel{x}(1-\frac{2}{x})} = \frac{1}{1} = 1 \right\} = [\ln 1] = 0$

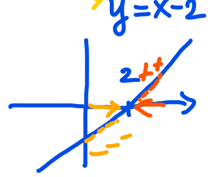
$\lim_{x\to-\infty} f(x) = \left\{ \lim_{x\to-\infty} \frac{x}{x-2} = 1 \quad \frac{2}{\infty} = \frac{2}{-\infty} = 0 \right\} = [\ln 1] = 0$

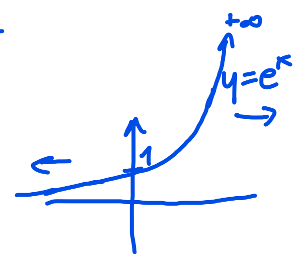
$y=0$ - as. pozioma w $+\infty$ i $-\infty$



$e^{\boxed{\frac{1}{x-2}}}$

Zadanie 2. Wyznaczyć asymptoty wykresu funkcji $f(x) = x + e^{\frac{1}{x-2}}$





$D = (-\infty, 2) \cup (2, +\infty)$

• As. pionowe

$\lim_{x\to 2^-} f(x) = \left[2 + e^{\frac{1}{0^-}}\right] = \left[2 + e^{-\infty}\right] = 2 + 0 = 2$

$\lim_{x\to 2^+} f(x) = \left[2 + e^{\frac{1}{0^+}}\right] = \left[2 + e^{+\infty}\right] = [2 + \infty] = +\infty$

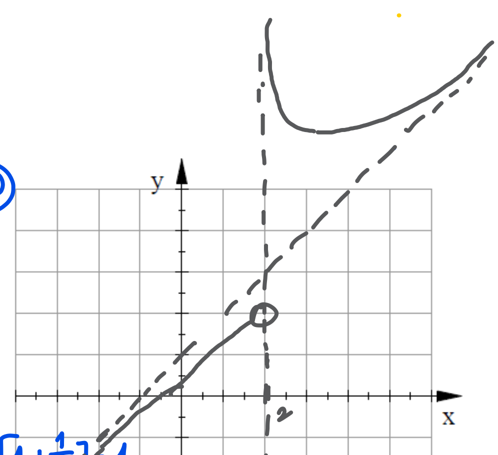
$x=2$ - as. pionowa prawostronna

• As. poziome

$\lim_{x\to+\infty} f(x) = \left[\infty + e^{\frac{1}{\infty - 2}}\right] = \left[\infty + e^{\frac{1}{\infty}}\right] = \left[\infty + e^0\right] = +\infty$

$\lim_{x\to-\infty} f(x) = \left[-\infty + e^{\frac{1}{-\infty}}\right] = -\infty$

brak as. poziomych



$\frac{1}{-\infty}$

• As. ukośne (w $+\infty$)

$y = ax + b$

$a = \lim_{x\to+\infty \, (-\infty)} \frac{f(x)}{x} = \lim_{x\to+\infty} \frac{x + e^{\frac{1}{x-2}}}{x} = \left[\frac{\infty}{\infty}\right] = \lim_{x\to+\infty \, (-\infty)} \left(1 + \frac{e^{\frac{1}{x-2}}}{x}\right) = \left[1 + \frac{1}{\infty}\right] = 1$

$b = \lim_{x\to+\infty \, (-\infty)} (f(x) - x) = \lim_{x\to+\infty \, (-\infty)} e^{\frac{1}{x-2}} = [e^0] = 1$

$y = x+1$ - as. ukośna w $+\infty$ i w $-\infty$

Zadanie domowe $f(x)=\frac{x+4}{\ln x}$

Zadanie 3. Wyznaczyć asymptoty wykresu funkcji $f(x)=\frac{x+4}{1-\ln x}$

$1-\ln x=0$
$\ln x=1$
$x=e$

$D: x>0 \wedge 1-\ln x\neq 0$
$x\neq e$

$D=(0,e)\cup(e,+\infty)$

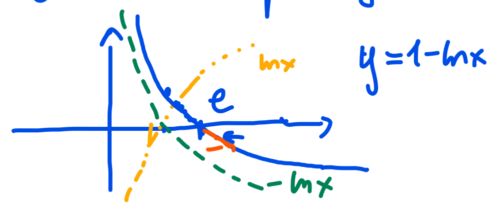
- As. pionowe

$\lim\limits_{x\to 0^+} f(x)=\lim\limits_{x\to 0^+}\frac{x+4}{1-\ln x}=\left[\frac{4}{1-\ln 0^+}\right]=\left[\frac{4}{1-(-\infty)}\right]=\left[\frac{4}{\infty}\right]=0\to$ brak as. pionowej

$\lim\limits_{x\to e^+} f(x)=\left[\frac{e+4}{0^-}\right]=-\infty$

$\lim\limits_{x\to e^-} f(x)=\left[\frac{e+4}{0^+}\right]=+\infty$

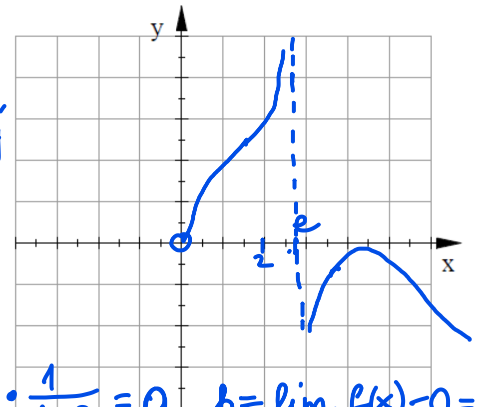
$\Big\}\ x=e$ as. pionowa obustronna



- As. poziome

$\lim\limits_{x\to+\infty} f(x)=\left[\frac{\infty}{1-\ln\infty}\right]=\left[\frac{\infty}{-\infty}\right]\overset{H}{=}\lim\limits_{x\to+\infty}\frac{(x+4)'}{(1-\ln x)'}=$

$=\lim\limits_{x\to+\infty}\frac{1}{-\frac{1}{x}}=\lim\limits_{x\to+\infty}(-x)=-\infty$ – brak as. poziomej



- As. ukośne w $+\infty$

$a=\lim\limits_{x\to+\infty}\frac{f(x)}{x}=\lim\limits_{x\to+\infty}\frac{\frac{x+4}{1-\ln x}}{x}=\lim\limits_{x\to\infty}\frac{x+4}{1-\ln x}\cdot\frac{1}{x}=$

$=\lim\limits_{x\to+\infty}\left(\frac{x+4}{x}\right)\cdot\frac{1}{1-\ln x}=\lim\limits_{x\to+\infty}\left(1+\frac{4}{x}\right)\cdot\frac{1}{1-\ln x}=1\cdot\frac{1}{1-\infty}=0$

$b=\lim\limits_{x\to+\infty} f(x)-0=-\infty$

brak as. ukośnej

Zadanie 4. Wyznaczyć asymptoty wykresu funkcji $f(x)=\frac{x-2}{e^{x-1}-1}$

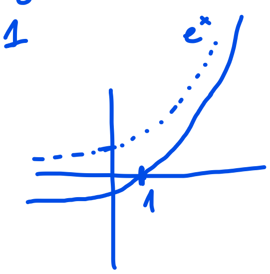
$D: e^{x-1}-1\neq 0$

$e^{x-1}-1=0 \Leftrightarrow e^{x-1}=e^0$
$x-1=0$
$x=1$

$D=(-\infty,1)\cup(1,+\infty)$

$\lim\limits_{x\to 1^-} f(x)=\left[\frac{1-2}{e^0-1}\right]=\left[\frac{-1}{0^-}\right]=+\infty$

$y=e^{x-1}-1$
$e^x\to[1,-1]$



$\lim\limits_{x\to 1^+} f(x)=\left[\frac{-1}{0^+}\right]=-\infty$

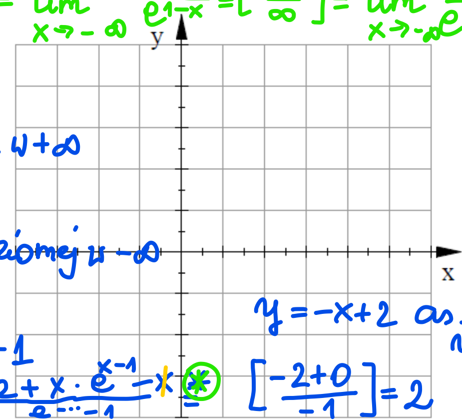
$x=1$ – as. pionowa obustronna

(*) $\lim\limits_{x\to-\infty} x e^{x-1}=[-\infty\cdot e^{-\infty}]=[\infty\cdot 0]=$
$=\lim\limits_{x\to-\infty}\frac{x}{e^{1-x}}=\left[\frac{-\infty}{\infty}\right]\overset{H}{=}\lim\limits_{x\to-\infty}\frac{1}{e^{1-x}\cdot(-1)}=\left[\frac{1}{-\infty}\right]$
$=0$

$\lim\limits_{x\to+\infty} f(x)=\lim\limits_{x\to+\infty}\frac{x-2}{e^{x-1}-1}=\left[\frac{\infty}{e^{\infty}-1}\right]=\left[\frac{\infty}{\infty}\right]\overset{H}{=}$

$=\lim\limits_{x\to+\infty}\frac{1}{e^{x-1}}=\left[\frac{1}{e^{\infty}}\right]=0\to y=0$ as. pozioma w $+\infty$

$\lim\limits_{x\to-\infty} f(x)=\left[\frac{-\infty}{e^{-\infty}-1}\right]=\left[\frac{-\infty}{-1}\right]=+\infty\to$ brak. as. poziomej w $-\infty$



w $-\infty$: $a=\lim\limits_{x\to-\infty}\frac{f(x)}{x}=\lim\limits_{x\to-\infty}\frac{x-2}{x}\cdot\frac{1}{e^{x-1}-1}=\left[1\cdot\frac{1}{-1}\right]=-1$

$y=-x+2$ as. ukośna w $-\infty$

$b=\lim\limits_{x\to-\infty}(f(x)+x)=\lim\limits_{x\to-\infty}\left(\frac{x-2}{e^{x-1}-1}+x\right)=\lim\frac{\cancel{x}-2+x\cdot e^{x-1}-\cancel{x}}{e^{x-1}-1}\overset{(*)}{=}\left[\frac{-2+0}{-1}\right]=2$