20 Zastosowania geometryczne całki oznaczonej

Całki nieoznaczone: szybka powtórka

1) Określić metodę rozwiązania podanych całek:

a) $\int \frac{2-\ln x}{x\cdot(3\ln^2 x+\ln x)}dx$ $\longrightarrow$ $\begin{cases} t=\ln x \\ dt=\frac{1}{x}dx \end{cases}$ $\int \frac{2-t}{3t^2+t}dt$ $\to$ całka wymierna

b) $\int \sin^3 2x dx$ $= \int \sin 2x\cdot\sin^2 2x\, dx = \int \sin 2x\cdot(1-\cos^2 2x)dx = -\frac{1}{2}\int(1-t^2)dt$

$t=\cos 2x,\ dt=-2\sin 2x dx,\ -\frac{1}{2}dt=\sin 2x dx$

c) $\int \frac{e^{2x}}{\sqrt{4-e^{2x}}}dx$ $= \begin{cases} t=e^{2x} \\ dt=2e^{2x}dx \end{cases} = \frac{1}{2}\int\frac{1}{\sqrt{4-t}}dt = \begin{cases} u=4-t \\ du=-dt \end{cases} = -\frac{1}{2}\int\frac{1}{\sqrt{u}}du$

$= -\sqrt{u}+C = -\sqrt{4-t}+C = -\sqrt{4-e^{2x}}+C$

d) $\int \frac{e^x}{\sqrt{4-e^{2x}}}dx$ $= t = ?$

e) $\int (x^2+1)e^{3x}dx$ $\to$ całkowanie przez części

$f(x)=x^2+1 \qquad g'(x)=e^{3x}$

f) $\int x^3\sin(x^2)dx$ $= \int x\cdot x^2\cdot\sin(x^2)dx = \begin{cases} t=x^2 \\ dt=2xdx \\ \frac{1}{2}dt=xdx \end{cases} = \frac{1}{2}\int t\sin t\, dt$ $\to$ całkowanie przez części

g) $\int \frac{\sin x\cos x}{(\sin^2 x+4\sin x+10)}dx$ $= \begin{cases} t=\sin x \\ dt=\cos x dx \end{cases} \to \int\frac{t\, dt}{t^2+4t+10}$ $\to$ całka wymierna

h) $\int \ln(x^2+1)dx$ $= \begin{cases} f(x)=\ln(x^2+1) & g'(x)=1 \end{cases}$

i) $\int \frac{x}{x^2+4x+12}dx$ $\longrightarrow$ $\Delta = ?$

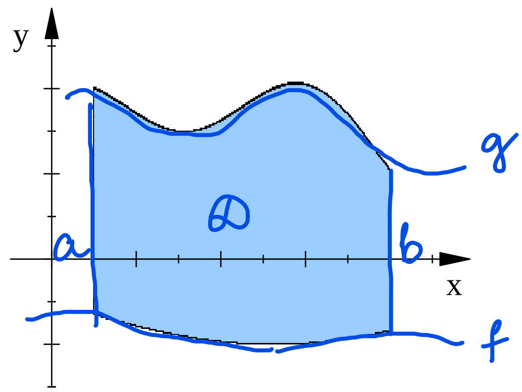
Twierdzenie. Jeśli zbiór $D \subset \mathbb{R}^2$ jest **obszarem normalnym względem osi Ox**, tzn. jest postaci

$$D = \{(x, y) \in \mathbb{R}^2 : x \in [a, b] \wedge f(x) \leq y \leq g(x)\}$$

gdzie f i g są funkcjami ciągłymi na przedziale $[a, b]$, to

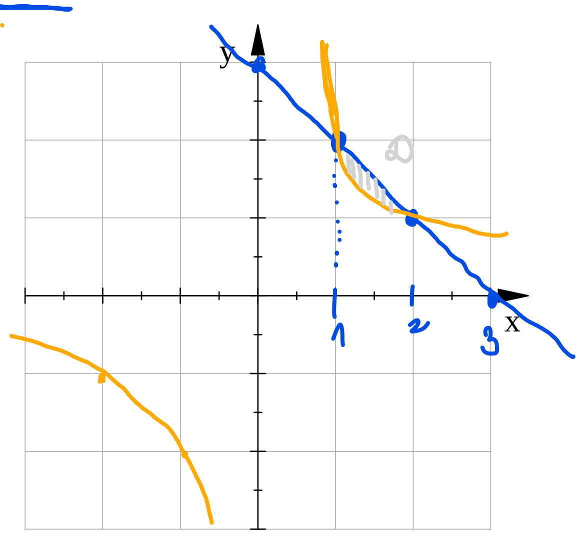
$$|D| = \int_a^b (g(x) - f(x))dx.$$

górny brzeg



2) Obliczyć pole figury ograniczonej krzywymi:

a) $y = \frac{2}{x}$, $y = 3 - x$



Wyznaczamy punkty przecięcia krzywych:

$$\begin{cases} y = \frac{2}{x} \\ y = 3-x \end{cases}$$

$$\frac{2}{x} = 3 - x \quad | \cdot x$$

$$2 = 3x - x^2$$

$$x^2 - 3x + 2 = 0$$

$$\Delta = 9 - 8 = 1$$

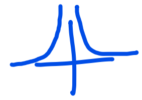
$$x_1 = \frac{3-1}{2} = 1 \qquad x_2 = 2$$

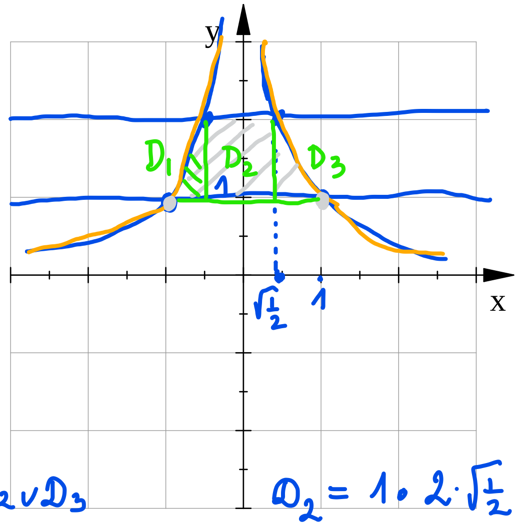
$$|D| = \int_1^2 \left(3 - x - \frac{2}{x}\right)dx =$$

$$= \left[3x - \frac{x^2}{2} - 2\ln|x|\right]_1^2 = 3 \cdot 2 - \frac{4}{2} - 2\ln 2 - \left(3 - \frac{1}{2} - 2\ln 1\right) =$$

$$= 6 - 2 - 2\ln 2 - 3 + \frac{1}{2} = \frac{3}{2} - 2\ln 2$$

b) $y=\frac{1}{x^2}, y=1, y=2$





$$\begin{cases} y=\frac{1}{x^2} \\ y=1 \end{cases}$$

$\frac{1}{x^2}=1$

$x^2=1$

$x=-1 \vee x=1$

$$\begin{cases} y=\frac{1}{x^2} \\ y=2 \end{cases}$$

$\frac{1}{x^2}=2$

$x^2=\frac{1}{2}$

$x=\sqrt{\frac{1}{2}} \vee x=-\sqrt{\frac{1}{2}}$

$D=D_1\cup D_2\cup D_3$ $\quad D_2=1\cdot 2\cdot\sqrt{\frac{1}{2}}=2\cdot\frac{\sqrt{2}}{2}=\sqrt{2}$

D_1 i D_3 są symetryczne względem osi Oy (funkcje parzyste)

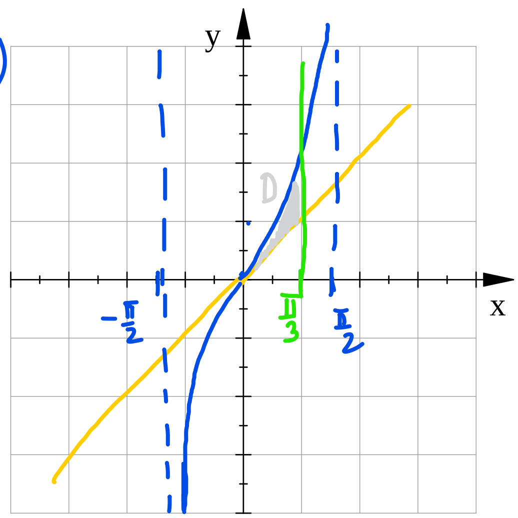
$|D_1|=|D_3|$

$$|D_3|=\int_{\frac{\sqrt{2}}{2}}^{1}\left(\frac{1}{x^2}-1\right)dx=\left[-\frac{1}{x}-x\right]_{\frac{\sqrt{2}}{2}}^{1}=-1-1-\left(-\frac{1}{\frac{\sqrt{2}}{2}}-\frac{\sqrt{2}}{2}\right)=-2+\sqrt{2}+\frac{\sqrt{2}}{2}$$

$$|D|=|D_1|+|D_2|+|D_3|=2\cdot\left(-2+\frac{3}{2}\sqrt{2}\right)+\sqrt{2}=-4+3\sqrt{2}+\sqrt{2}=-4+4\sqrt{2}=4(\sqrt{2}-1)$$

c) $y=\operatorname{tg}x, y=x, x=\frac{\pi}{3}$

$x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$



$$|D|=\int_0^{\frac{\pi}{3}}(\operatorname{tg}x-x)\,dx=$$

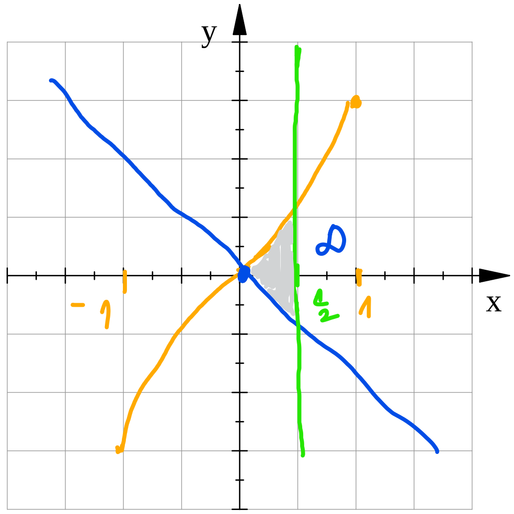
$$=-\int_0^{\frac{\pi}{3}}\frac{-\sin x}{\cos x}\,dx-\int_0^{\frac{\pi}{3}}x\,dx$$

$$\left\{\int\frac{f'}{f}=\ln|f|+C\right\}$$

$$|D|=-\left[\ln|\cos x|\right]_0^{\frac{\pi}{3}}-\left[\frac{x^2}{2}\right]_0^{\frac{\pi}{3}}=-\left(\ln|\underbrace{\cos\tfrac{\pi}{3}}_{\frac{1}{2}}|-\ln|\underbrace{\cos 0}_{1}|\right)-\left(\frac{\left(\frac{\pi}{3}\right)^2}{2}-0\right)=$$

$$=-\underbrace{\ln\frac{1}{2}}_{\ln 2^{-1}}+\underbrace{\ln 1}_{0}-\frac{\pi^2}{18}=\ln 2-\frac{\pi^2}{18}$$

d) $y = \arcsin x, y = -x, x = \frac{1}{2}$

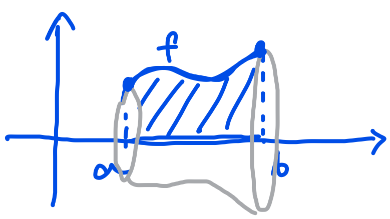


$$|D| = \int_0^{\frac{1}{2}} \left(\arcsin x - (-x)\right) dx = \dots$$

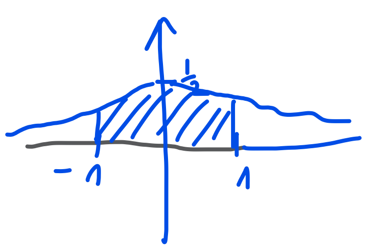
↓

całkowanie przez części

3. Obliczyć objętość bryły powstałej z obrotu dookoła osi OX zbioru ograniczonego krzywymi $y = \frac{1}{\sqrt{x^2+4}}$, $y=0$, $x=1$, $x=-1$



$$|V| = \pi \int_a^b f^2(x)\,dx$$



$$y = \frac{1}{\sqrt{x^2+4}}, \; x \in R$$

$$|V| = \pi \int_{-1}^{1} \frac{1}{x^2+4}\,dx = \pi \left[\frac{1}{2}\operatorname{arctg}\frac{x}{2}\right]_{-1}^{1} =$$

$$= \frac{\pi}{2}\left(\operatorname{arctg}\frac{1}{2} - \operatorname{arctg}\left(-\frac{1}{2}\right)\right) = \frac{\pi}{2}\left(\operatorname{arctg}\frac{1}{2} + \operatorname{arctg}\frac{1}{2}\right) =$$

$$= \frac{\pi}{2} \cdot 2\operatorname{arctg}\frac{1}{2} = \pi \operatorname{arctg}\frac{1}{2}$$