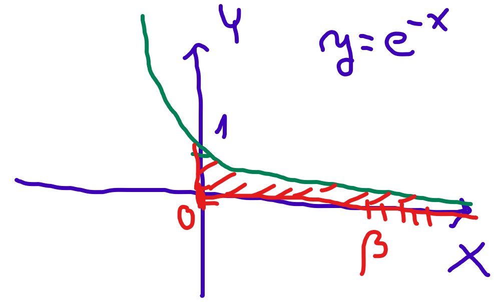
CAŁKI NIEWŁAŚCIWE



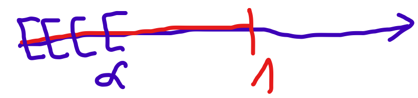
(1) $$\int_0^{+\infty} e^{-x}\,dx = \lim_{\beta\to+\infty} \underbrace{\int_0^{\beta} e^{-x}\,dx}_{\text{całka oznaczona}} = (*)$$

$$\int e^{-x}\,dx = -e^{-x}+C$$

$$(*) = \lim_{\beta\to+\infty}\left[-e^{-x}\right]\Big|_0^{\beta} = \lim_{\beta\to+\infty}\left(-e^{-\beta}+e^{0}\right) = \{-0+1\} = 1$$ – CAŁKA ZBIEŻNA

(2) $$\int_{-\infty}^{1} \frac{1}{x^2-4x+5}\,dx = \lim_{\alpha\to-\infty} \underbrace{\int_{\alpha}^{1} \frac{1}{x^2-4x+5}\,dx}_{\text{całka oznaczona}} = (*)$$

$\Delta<0$



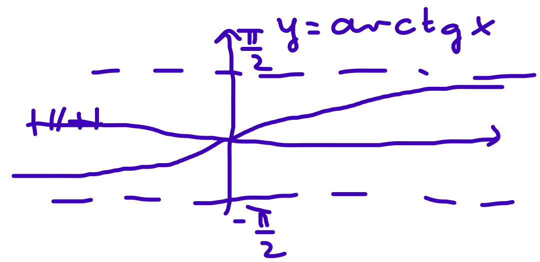
$$\int \frac{1}{x^2-4x+5}\,dx = \int \frac{1}{(x^2-4x+4)+1}\,dx = \int \frac{1}{(x-2)^2+1}\,dx = \left\{\begin{matrix} x-2=t \\ dx=dt \end{matrix}\right\} =$$

$$= \int \frac{1}{t^2+1}\,dt = \operatorname{arctg} t + C = \operatorname{arctg}(x-2)+C$$

$$(*)\ \lim_{x\to-\infty}\left[\operatorname{arctg}(x-2)\right]\Big|_{\alpha}^{1} = \lim_{x\to-\infty}\left(\operatorname{arctg}(-1)-\operatorname{arctg}(\alpha-2)\right) =$$

$$= \left\{-\frac{\pi}{4}-\operatorname{arctg}(-\infty)\right\} = -\frac{\pi}{4}-\left(-\frac{\pi}{2}\right) = \frac{\pi}{4}$$

↑ całka zbieżna

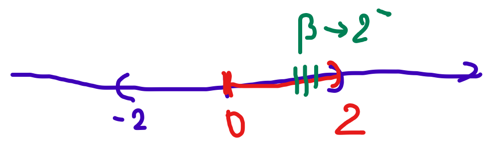


(3) $$\int_{-\infty}^{+\infty} \frac{1}{x^2-4x+5}\,dx = \int_{-\infty}^{1} \frac{1}{x^2-4x+5}\,dx + \int_{1}^{+\infty} \frac{1}{x^2-4x+5}\,dx$$

(4) $$\int_0^{2} \frac{x}{\sqrt{4-x^2}}\,dx = \lim_{\beta\to2^-} \int_0^{\beta} \frac{x}{\sqrt{4-x^2}}\,dx = (*)$$

$$f(x) = \frac{x}{\sqrt{4-x^2}}$$

$$D_f = \{x\in\mathbb{R} : 4-x^2>0\} = (-2,2)$$



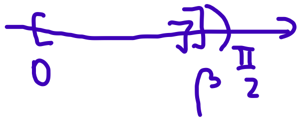
$$-\frac{1}{2}\int \frac{-2x}{\sqrt{4-x^2}}\,dx = -\frac{1}{2}\cdot 2\sqrt{4-x^2}+C = -\sqrt{4-x^2}+C$$

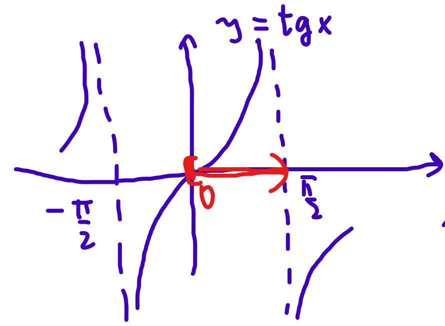
WZÓR:
$$\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+C$$

$$(*) = \lim_{\beta \to 2^-} \left[-\sqrt{4-x^2}\right]\Big|_0^\beta = \lim_{\beta \to 2^-} \left(-\sqrt{4-\beta^2}+2\right) = \{-2+2\} = 0$$

całka zbieżna

(5) $$\int_0^{\frac{\pi}{2}} \operatorname{tg}x\,dx = \lim_{\beta \to \frac{\pi}{2}^-} \underbrace{\int_0^\beta \operatorname{tg}x\,dx}_{} = (*)$$





$$\int \operatorname{tg}x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x| + C$$

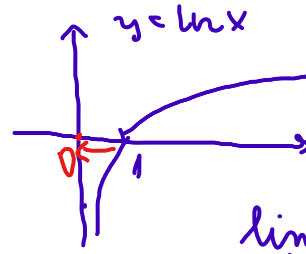
WZÓR
$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$(*) = \lim_{\beta \to \frac{\pi}{2}^-} \left[-\ln|\cos x|\right]\Big|_0^\beta =$$

$$= \lim_{\beta \to \frac{\pi}{2}^-} \left[-\ln|\cos\beta| + \underbrace{\ln|\cos 0|}_{=0}\right] =$$

$\cos 0 = 1$
$\ln 1 = 0$

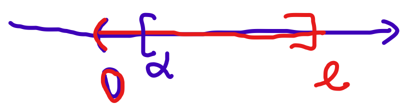
$$= \{-\ln(0^+)\} = \{-(-\infty)\} = +\infty$$ – CAŁKA ROZBIEŻNA



$$\lim_{x \to 0^+} \ln x = -\infty$$

(6) $$\int_0^e \ln x\,dx = \lim_{\alpha \to 0^+} \int_\alpha^e \ln x\,dx = (*)$$

$f(x) = \ln x$
$D_f = (0, +\infty)$



$$\int 1\cdot\ln x\,dx = \left\{\begin{matrix} f(x) = \ln x & g'(x) = 1 \\ f'(x) = \frac{1}{x} & g(x) = x \end{matrix}\right\} = x\ln x - \int \frac{1}{x}\cdot x\,dx = x\ln x - x + C$$

$$(*) = \lim_{\alpha \to 0^+} \left[x\ln x - x\right]\Big|_\alpha^e = \lim_{\alpha \to 0^+} \left[e\cdot\underbrace{\ln e}_{=1} - e - \alpha\ln\alpha + \alpha\right] =$$

$$= \lim_{\alpha \to 0^+} \alpha(1-\ln\alpha) = \overset{S.N}{\{0\cdot(+\infty)\}} = \lim_{\alpha \to 0^+} \frac{1-\ln\alpha}{\frac{1}{\alpha}} = \overset{S.N}{\left\{\frac{+\infty}{+\infty}\right\}} \overset{H}{=}$$

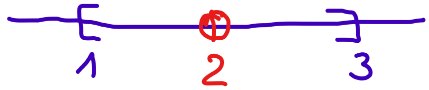
$$= \lim_{\alpha \to 0^+} \frac{-\frac{1}{\alpha}}{-\frac{1}{\alpha^2}} = \lim_{\alpha \to 0^+} \frac{1}{\alpha}\cdot\alpha^2 = \lim_{\alpha \to 0^+} \alpha = 0$$

całka zbieżna

(7) $\displaystyle\int_1^3 \frac{x}{4-x^2}\,dx = \int_1^2 \frac{x}{4-x^2}\,dx + \int_2^3 \frac{x}{4-x^2}\,dx$

$f(x) = \dfrac{x}{4-x^2}$

$D_f = \mathbb{R} \setminus \{2, -2\}$



(8) $\displaystyle\int_0^{+\infty} \ln x\,dx = \int_0^e \ln x\,dx + \int_e^{+\infty} \ln x\,dx$

$\displaystyle\int_0^1 \frac{1}{x^2}\cdot e^{\frac{1}{x}}\,dx$

$\displaystyle\int_1^3 \frac{1}{x-1}\,dx$