28 Liczby zespolone - powtórzenie

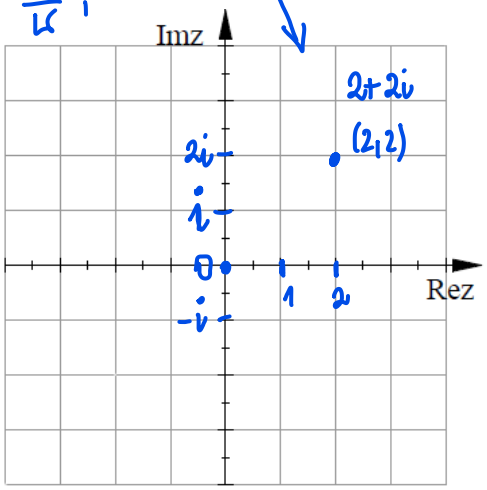
(x,y) $\qquad \varphi = \operatorname{Arg} z$

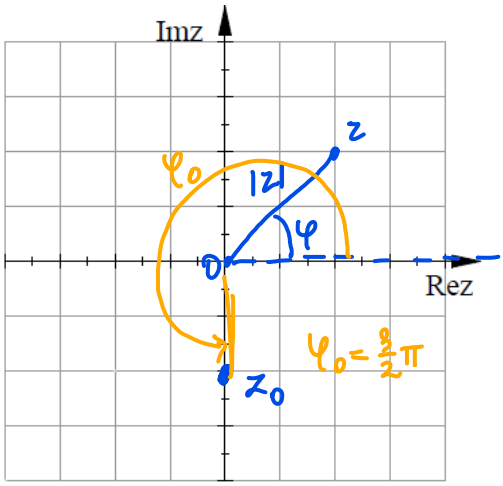
$z = x + iy$ $\qquad z = |z|\,(\cos\varphi + i\sin\varphi)$ $\qquad z = |z| \cdot e^{i\varphi}$

Postać kartezjańska $\qquad$ Postać trygonometryczna $\qquad$ Postać wykładnicza

$z+w$
$z-w$
$z\cdot w$
$\frac{z}{w}$,

2^n , $\sqrt[n]{z}$





Zadanie 1. Niech $z = 3 - i$. Oblicz

a) $\operatorname{Im} z + \operatorname{Re}(\frac{\overline{z+2i}}{z-1}) =$ $-1 + \operatorname{Re}\left(\frac{7+i}{5}\right) = -1 + \frac{7}{5} = \frac{2}{5}$

$\operatorname{Im} z = -1$ $\qquad \bar{z} = 3+i$ $\qquad \operatorname{Im}\bar{z} = 1$

$\overline{z+2i} = \overline{3-i+2i} = \overline{3+i} = 3-i$ $\qquad z-1 = 3-i-1 = 2-i$

$\frac{\overline{z+2i}}{z-1} = \frac{3-i}{2-i} \cdot \frac{2+i}{2+i} = \frac{6+3i-2i-i^2}{4-i^2} = \frac{6+i+1}{5} = \frac{7+i}{5}$ $\qquad (a-b)\cdot(a+b) = a^2-b^2$

b) $\left|\frac{\overline{z+2i}}{z-1}\right| =$ $\left|\frac{7+i}{5}\right| = \sqrt{\left(\frac{7}{5}\right)^2 + \left(\frac{1}{5}\right)^2} = \sqrt{\frac{49}{25} + \frac{1}{25}} = \sqrt{\frac{50}{25}} = \sqrt{2}$ $\qquad |z| = \sqrt{x^2+y^2}$

1 sposób

2 sposób $\qquad \left|\frac{w_1}{w_2}\right| = \frac{|w_1|}{|w_2|}$ własność

$\left|\frac{z+2i}{z-1}\right| = \frac{|z+2i|}{|z-1|} = \frac{|3-i|}{|2-i|} = \frac{\sqrt{9+1}}{\sqrt{4+1}} = \frac{\sqrt{10}}{\sqrt{5}} = \sqrt{2}$

Zadanie 2. Oblicz i zapisz w postaci kartezjańskiej: $(-1+\sqrt{3}i)^{35}$.

$z = |z|(\cos\varphi + i\sin\varphi)$

- $z = -1+\sqrt{3}i$ $\quad |z| = \sqrt{1+3} = 2$

$$\begin{cases} \cos\varphi = \frac{x}{|z|} \\ \sin\varphi = \frac{y}{|z|} \end{cases} \qquad \begin{cases} \cos\varphi = \frac{-1}{2} & - \\ \sin\varphi = \frac{\sqrt{3}}{2} & + \end{cases} \quad \text{ćw. II}$$

$\text{Arg}\, z \in [0, 2\pi)$

$\sin\varphi = \frac{\sqrt{3}}{2} \qquad \sin\varphi = \sin\frac{\pi}{3}$

I ćw: $\varphi = \frac{\pi}{3} + 2k\pi$ $\quad \vee$ II ćw: $\varphi = \pi - \frac{\pi}{3} + 2k\pi$

$\varphi = \frac{2}{3}\pi + 2k\pi$

$\frac{70\pi}{3} = 22\pi + \frac{4}{3}\pi$, $11 \cdot 2\pi$

- $z = 2 \cdot \left(\cos\frac{2}{3}\pi + i\sin\frac{2}{3}\pi\right)$

- $z^{35} = 2^{35}\left(\cos\left(35 \cdot \frac{2\pi}{3}\right) + i\sin\left(35\cdot\frac{2\pi}{3}\right)\right) = 2^{35}\left(\cos\frac{70\pi}{3} + i\sin\frac{70\pi}{3}\right) =$

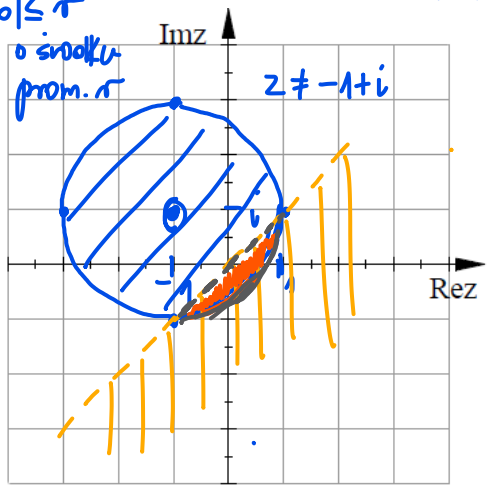
$= 2^{35}\left(\cos\left(11\cdot 2\pi + \frac{4}{3}\pi\right) + i\sin\left(11\cdot 2\pi + \frac{4}{3}\pi\right)\right) \overset{\text{okresowość}}{=} 2^{35}\left(\cos\frac{4}{3}\pi + i\sin\left(\pi + \frac{\pi}{3}\right)\right) =$ (π i ćw)

$= 2^{35}\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = -2^{35}\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2^{34}(1+\sqrt{3}i)$

Zadanie 3. Naszkicuj na płaszczyźnie zespolonej zbiory:

$A = \{z \in \mathbb{C} : 0 < |z+1-i| \le 2 \ \wedge \ \text{Im}\, z < \text{Re}\, z\}$ $\quad \hookrightarrow z_0 = -1+i$

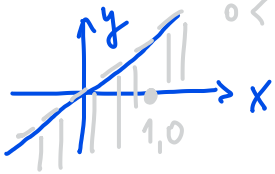
$|z - z_0| \le r$ koło o środku z_0 i promieniu r



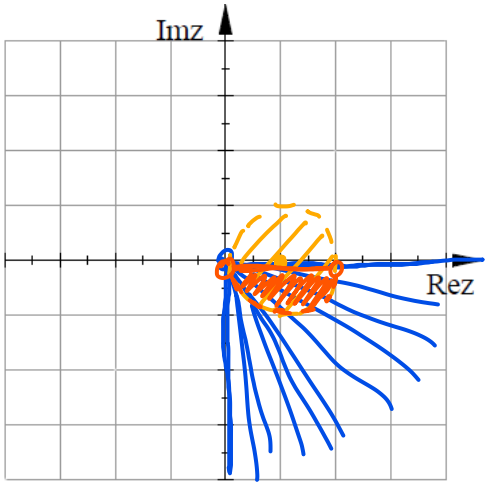
$\text{Im}\, z < \text{Re}\, z$

$z = x + iy \Rightarrow (\text{Im}\, z = y \ \text{i} \ \text{Re}\, z = x)$

$\text{Im}\, z < \text{Re}\, z \iff y < x$



$B = \{z \in \mathbb{C} : \frac{3\pi}{2} < \text{Arg}\, z \le 2\pi \ \wedge \ \overline{z}\cdot z < \text{Re}(2z)\}$



$\text{Arg}\, 0 = 0$

$\overline{z}\cdot z < \text{Re}(2z)$

$z = x + iy$

- $\overline{z}\cdot z = (x-iy)(x+iy) = x^2 - i^2y^2 = x^2 + y^2$
- $\text{Re}(2z) = \text{Re}(2x + 2iy) = 2x$

$x^2 + y^2 < 2x \iff x^2 - 2x + y^2 < 0$

$x^2 - 2x + 1 + y^2 < 1$

$S = (1,0) \quad r = 1 \qquad (x-1)^2 + y^2 < 1$

Zadanie 4 Rozwiąż równania w dziedzinie zespolonej:

a) $z^2 - 2z + 3 = 0$ b) $iz^2 - (1+2i)z + 1 = 0$ c) $z^3 = 4iz$.

a) $\Delta = 4 - 12 = -8$

$\sqrt{\Delta} = \{+2\sqrt{2}i, -2\sqrt{2}i\}$ (δ_1, δ_2)

$z = \frac{-b+\delta_1}{2a} \vee z = \frac{-b+\delta_2}{2a}$

$z = \frac{2+2\sqrt{2}i}{2} \vee z = \frac{2-2\sqrt{2}i}{2}$

$z = 1+\sqrt{2}i \vee z = 1-\sqrt{2}i$

b) $iz^2 - (1+2i)z + 1 = 0$

$a = i \quad b = -(1+2i) \quad c = 1$

$\Delta = (-(1+2i))^2 - 4\cdot i = 1 + 4i + 4i^2 - 4i = -3$

$\sqrt{\Delta} = \{-\sqrt{3}i, \sqrt{3}i\}$

$z = \frac{1+2i-\sqrt{3}i}{2i} \cdot \frac{i}{i} \vee z = \frac{1+2i+\sqrt{3}i}{2i} \cdot \frac{i}{i}$

$z = \frac{i+2i^2-\sqrt{3}i^2}{2i^2} \vee z = \frac{i+2i^2+\sqrt{3}i^2}{2i^2}$

$z = \frac{i-2+\sqrt{3}}{-2} \vee z = \frac{i-2-\sqrt{3}}{-2}$

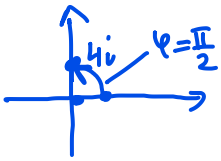
$z = 1 - \frac{\sqrt{3}}{2} - \frac{i}{2} \vee z = 1 + \frac{\sqrt{3}}{2} - \frac{i}{2}$

c) $z^3 - 4iz = 0$

$z(z^2 - 4i) = 0$

$z = 0 \vee z^2 - 4i = 0$

$z^2 = 4i$



$n = 2 \quad \sqrt{4i}$

wykorzystujemy postać trygon.

$a = 4i = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$w_k = \sqrt{4}\left(\cos\frac{\frac{\pi}{2}+2k\pi}{2} + i\sin\left(\frac{\pi}{4}+k\pi\right)\right)$

$k = 0, 1$

$w_0 = 2\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 2\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \sqrt{2} + \sqrt{2}i$

$w_1 = 2\left(\cos\left(\frac{\pi}{4}+\pi\right) + i\sin\left(\frac{\pi}{4}+\pi\right)\right) = 2\left(-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right) = -\sqrt{2} - \sqrt{2}i$

Odp. $z = 0 \vee z = \sqrt{2}+\sqrt{2}i \vee z = -\sqrt{2}-\sqrt{2}i$

Tw. Jeśli $z = |z|(\cos\varphi + i\sin\varphi)$ to pierwiastki n -tego stopnia z liczby z opisane są wzorem:

$$w_k = \sqrt[n]{|z|}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

d) $z^4 + 16 = 0 \iff z^4 = -16 \qquad \sqrt[4]{-16} = ?$