Całki - powtórka

Zadania

Zadanie 1 Oblicz całki:

1.1. $\int \frac{1}{x\sqrt{x^4-1}}dx$

$\int \frac{e^x}{e^{2x}+2e^x+2}dx$

$\int \sin^3(x)\sqrt{\cos(x)}dx$

$\int \frac{\ln(\ln x)}{x\ln x}dx$

$\int \ln(x^2+1)dx$

1.6. $\int (\ln x)^2 dx$

Zadanie 2 Zbadaj zbieżność całek:

2.1. $\int_1^\infty \frac{1}{x\sqrt{x^4-1}}dx$

$\int_0^\infty \frac{e^x}{e^{2x}+2e^x+2}dx$

$\int_0^{\pi/2} \sin^3(x)\sqrt{\cos(x)}dx$

$\int_e^\infty \frac{\ln(\ln x)}{x\ln x}dx$

$\int_0^\infty \ln\left(\frac{x^2+1}{x^2}\right)dx$

2.6. $\int_0^1 (\ln x)^2 dx$

$1 \leftarrow 2$

2.1 $$\underbrace{\int_1^\infty \frac{1}{x\sqrt{x^4-1}}dx}_{\text{całka zbieżna}} = \underbrace{\int_1^2 \frac{1}{x\sqrt{x^4-1}}dx}_{\text{całka zbieżna II}} + \underbrace{\int_2^\infty \frac{1}{x\sqrt{x^4-1}}dx}_{\text{całka zbieżna I}} = \underbrace{\lim_{a\to 1^+}\int_a^2 \frac{1}{x\sqrt{x^4-1}}dx}_{A} + \underbrace{\lim_{b\to\infty}\int_2^b \frac{1}{x\sqrt{x^4-1}}dx}_{B} = \cancel{\frac12\arccos\frac14} + \frac{\pi}{4} - \cancel{\frac12\arccos\frac14} = \frac{\pi}{4}$$

$\cos(\alpha)=1 \quad \alpha\in[0,\pi] \Rightarrow \alpha=0$

$$A = \lim_{a\to 1^+}\left[\frac12\arccos\frac{1}{x^2}\right]\Big|_a^2 = \lim_{a\to 1^+}\left[\frac12\arccos\frac14 - \frac12\arccos\frac{1}{a^2}\right] = \left\{\frac12\arccos\frac14 - \frac12\arccos 1 = \frac12\arccos\frac14 - 0\right\} = \frac12\arccos\frac14$$

$\cos(\beta)=0 \quad \beta\in[0,\pi] \Rightarrow \beta=\frac{\pi}{2}$

$$B = \lim_{b\to+\infty}\left[\frac12\arccos\frac{1}{x^2}\right]_2^b = \lim_{b\to\infty}\left[\frac12\arccos\frac{1}{b^2} - \frac12\arccos\frac14\right] = \left\{\frac12\arccos\left(\frac{1}{\infty}\right) - \frac12\arccos\frac14 = \frac12\arccos 0 - \frac12\arccos\frac14 = \frac12\cdot\frac{\pi}{2} - \frac12\arccos\frac14\right\} = \frac{\pi}{4} - \frac12\arccos\frac14$$

1.1. $$\int\frac{1}{x\sqrt{x^4-1}}dx = \left\{\begin{array}{l} t=x^2 \\ dt = 2x\,dx \\ \frac12 dt = x\,dx\end{array}\right\} = \int\frac{x\,dx}{x^2\sqrt{x^4-1}} = \int\frac{\frac12 dt}{t\sqrt{t^2-1}} = \left\{\begin{array}{l} z=\frac1t \Rightarrow t=\frac1z \\ dt = -\frac{1}{z^2}dz\end{array}\right\} = \frac12\int\frac{-\frac{1}{z^2}dz}{\frac1z\sqrt{\frac{1}{z^2}-1}} = -\frac12\int\frac{z\,dz}{z^2\sqrt{\frac{1}{z^2}-1}} = -\frac12\int\frac{dz}{z\sqrt{\frac{1}{z^2}-1}} = -\frac12\int\frac{dz}{\sqrt{z^2\left(\frac{1}{z^2}-1\right)}} = -\frac12\int\frac{dz}{\sqrt{1-z^2}} = \frac12\arccos z + C = \frac12\arccos\frac1t + C = \frac12\arccos\frac{1}{x^2}+C$$

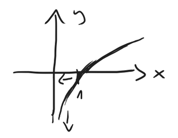
2.6. $$\underbrace{\int_0^1(\ln x)^2dx}_{\text{całka zbieżna}} = \lim_{a\to0^+}\int_a^1(\ln x)^2dx = \lim_{a\to0^+}\left[x(\ln x)^2 - 2x\ln x + 2x\right]\Big|_a^1 = \lim_{a\to0^+}\left[\underbrace{1(\ln 1)^2}_{0} - \underbrace{2\cdot1\cdot\ln1}_{0} + 2 - \left(a(\ln a)^2 - 2a\ln a + 2a\right)\right] = \lim_{a\to0^+}\left[2 - \boxed{a(\ln a)^2} + \boxed{2a\ln a} - 2a\right] = \left\{2 - 0\cdot(\ln 0^+)^2 + 2\cdot0\cdot\ln0^+ - 2\cdot0 = 2 - \underbrace{\boxed{0\cdot(-\infty)^2}}_{SN} + \underbrace{\boxed{2\cdot0\cdot(-\infty)}}_{SN} - 0\right\} = \{2-0+0-0\} = 2$$

$\ln 1 = 0$

$$\lim_{a\to0^+}a(\ln a)^2 = \{0\cdot\infty\} = \lim_{a\to0^+}\frac{(\ln a)^2}{\frac1a} = \left\{\frac{+\infty}{\frac{1}{0^+}} = \frac{+\infty}{+\infty}\right\} \overset{H}{=} \lim_{a\to0^+}\frac{2\ln a\cdot\frac1a}{-\frac{1}{a^2}} = \lim_{a\to0^+}\frac{2\ln a}{-\frac1a} = \left\{\frac{-\infty}{-\infty}\right\} \overset{H}{=} \lim_{a\to0^+}\frac{\frac2a}{\frac{1}{a^2}} = \lim_{a\to0^+}\frac{2}{\frac1a} = \lim_{a\to0^+}2a = 0$$

$0\cdot\infty$

$$\lim_{a\to0^+}2a\ln a = \{2\cdot0\cdot(-\infty)\} = \lim_{a\to0^+}\frac{2\ln a}{\frac1a} = \left\{\frac{-\infty}{+\infty}\right\} \overset{H}{=} \lim_{a\to0^+}\frac{\frac2a}{-\frac{1}{a^2}} = \lim_{a\to0^+}\frac{2}{-\frac1a} = \lim_{a\to0^+}(-2a) = 0$$

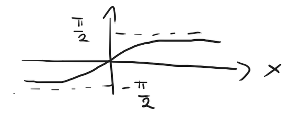


1.6. $$\int(\ln x)^2dx = \left\{\begin{array}{ll} f(x)=(\ln x)^2 & g'(x)=1 \\ f'(x)=2\cdot\ln x\cdot\frac1x & g(x)=x\end{array}\right\} = x\cdot(\ln x)^2 - \int 2\ln x\cdot\frac1x\cdot x\,dx = x(\ln x)^2 - \int 2\ln x\,dx = \left\{\begin{array}{ll} f(x)=\ln x & g'(x)=2 \\ f'(x)=\frac1x & g(x)=2x\end{array}\right\} = x(\ln x)^2 - \left[2x\ln x - \int\frac1x 2x\,dx\right] = x(\ln x)^2 - 2x\ln x + 2x + C$$

$\int 2dx$

(b) $$\int_0^\infty \frac{1}{x^2+4x+5}dx = \lim_{a\to+\infty}\int_0^a \frac{1}{x^2+4x+5}dx = \lim_{a\to+\infty}\operatorname{arctg}(x+2)\Big|_0^a = \lim_{a\to+\infty}\left(\operatorname{arctg}(a+2) - \operatorname{arctg}2\right) = \left\{\operatorname{arctg}(+\infty) - \operatorname{arctg}2\right\} = \frac{\pi}{2} - \operatorname{arctg}2$$

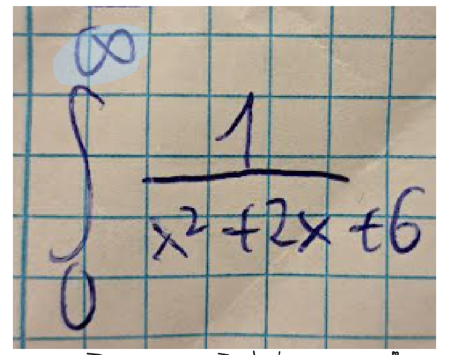
całka jest zbieżna



$x^2+4x+4 = (x+2)^2$

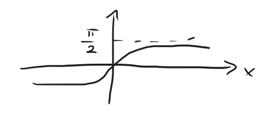
$x^2+4x+5 = (x+2)^2+1 \quad > 0$

$$\int\frac{1}{x^2+4x+5}dx = \int\frac{1}{(x+2)^2+1}dx = \left\{\begin{array}{l} t=x+2 \\ dt=dx\end{array}\right\} = \int\frac{1}{t^2+1}dt = \operatorname{arctg}t + C = \operatorname{arctg}(x+2)+C$$



$$= \lim_{a\to\infty}\int_0^a\frac{1}{x^2+2x+6}dx = \lim_{a\to\infty}\frac{\sqrt5}{5}\operatorname{arctg}\frac{x+1}{\sqrt5}\Big|_0^a = \lim_{a\to\infty}\frac{\sqrt5}{5}\left(\operatorname{arctg}\frac{a+1}{\sqrt5} - \operatorname{arctg}\frac{0+1}{\sqrt5}\right) = \left\{\frac{\sqrt5}{5}\left(\operatorname{arctg}(\infty) - \operatorname{arctg}\frac{1}{\sqrt5}\right)\right\} = \frac{\sqrt5}{5}\left(\frac{\pi}{2} - \operatorname{arctg}\frac{1}{\sqrt5}\right)$$

całka niewłaściwa zbieżna



$x^2+2x+6 = x^2+2x+1+5 = (x+1)^2+5 > 0$

$$\int\frac{1}{x^2+2x+6}dx = \int\frac{1}{(x+1)^2+5}dx = \left\{\begin{array}{l} t=x+1 \\ dt=dx\end{array}\right\} = \int\frac{1}{t^2+5}dt = \int\frac{1}{5\left(\frac{t^2}{5}+1\right)}dt = \frac15\int\frac{1\,dt}{\left(\frac{t}{\sqrt5}\right)^2+1} = \left\{\begin{array}{l} z=\frac{t}{\sqrt5}=\frac{1}{\sqrt5}\cdot t \\ dz=\frac{1}{\sqrt5}dt \\ \sqrt5 dz = dt\end{array}\right\} = \frac15\int\frac{\sqrt5 dz}{z^2+1} = \frac{\sqrt5}{5}\int\frac{dz}{z^2+1} = \frac{\sqrt5}{5}\operatorname{arctg}z + C = \frac{\sqrt5}{5}\operatorname{arctg}\left(\frac{t}{\sqrt5}\right)+C = \frac{\sqrt5}{5}\operatorname{arctg}\frac{x+1}{\sqrt5}+C$$