1 Granice ciągów liczbowych

$\frac{\infty}{\infty}$ $\frac{0}{0}$ ∞^0 1^∞ 0^0 $\infty-\infty$ $0\cdot\infty$

Do przypomnienia: własności potęg i pierwiastków, symbole oznaczone i nieoznaczone

1) Oblicz granice ciągów: $n, n^2, \sqrt{n} \to \infty$ $\quad \frac{1}{n}, \frac{1}{n^2}, \frac{1}{\sqrt{n}} \to 0$

a) $\lim\limits_{n\to\infty}(n-4n^2-5) = [\infty-\infty] = \lim\limits_{n\to\infty} n^2\left(\frac{1}{n}-4-\frac{5}{n^2}\right) = [\infty\cdot(0-4-0)] = -\infty$

b) $\lim\limits_{n\to\infty}\frac{2-3n}{n-1} = \left[\frac{-\infty}{\infty}\right] = \lim\limits_{n\to\infty}\frac{n\left(\frac{2}{n}-3\right)}{n\left(1-\frac{1}{n}\right)} = \left[\frac{0-3}{1-0}\right] = -3$

c) $\lim\limits_{n\to\infty}\frac{7-2n^4}{1+2n-4n^4} = \left[\frac{-\infty}{-\infty}\right] = \lim\limits_{n\to\infty}\frac{n^4\left(\frac{7}{n^4}-2\right)}{n^4\left(\frac{1}{n^4}+\frac{2}{n^3}-4\right)} = \left[\frac{0-2}{0-4}\right] = \frac{1}{2}$

d) $\lim\limits_{n\to\infty}\frac{7-2n}{1+2n-4n^4} = \lim\limits_{n\to\infty}\frac{n^4\left(\frac{7}{n^4}-\frac{2}{n^3}\right)}{n^4\left(\frac{1}{n^4}+\frac{2}{n^3}-4\right)} = \left[\frac{0-0}{-4}\right] = 0$

e) $\lim\limits_{n\to\infty}\frac{7-2n^4}{1+2n-4n^2} = \lim\limits_{n\to\infty}\frac{n^2\left(\frac{7}{n^2}-2n^2\right)}{n^2\left(\frac{1}{n^2}+\frac{2}{n}-4\right)} = \left[\frac{0-\infty}{-4}\right] = +\infty$

2) Oblicz granice ciągów: $\sqrt{n} \to \infty$ $\quad \sqrt{\infty} = \infty$

a) $\lim\limits_{n\to\infty}(\sqrt{2n+1}+\sqrt{2n+2}) = [\infty+\infty] = +\infty$

$(a-b)(a+b) = a^2-b^2$

$(\sqrt{x})^2 = x$ dla $x \geqslant 0$

b) $\lim\limits_{n\to\infty}(\sqrt{2n+1}-\sqrt{2n+2}) = [\infty-\infty] = \lim\limits_{n\to+\infty}\left(\sqrt{2n+1}-\sqrt{2n+2}\right)\cdot\frac{\sqrt{2n+1}+\sqrt{2n+2}}{\sqrt{2n+1}+\sqrt{2n+2}} =$

$= \lim\limits_{n\to\infty}\frac{\left(\sqrt{2n+1}\right)^2-\left(\sqrt{2n+2}\right)^2}{\sqrt{2n+1}+\sqrt{2n+2}} = \lim\limits_{n\to\infty}\frac{2n+1-2n-2}{\sqrt{2n+1}+\sqrt{2n+2}} = \left[\frac{-1}{\infty}\right] = 0$

c) $\lim\limits_{n\to\infty}\frac{1}{\sqrt{2n+1}+\sqrt{2n+2}} = \left[\frac{1}{\infty+\infty}\right] = 0$

$\lim\limits_{n\to\infty}\frac{1}{\sqrt{2n+1}-\sqrt{2n+2}}\cdot\frac{\sqrt{}+\sqrt{}}{\sqrt{}+\sqrt{}}$

d) $\lim\limits_{n\to\infty}\frac{\sqrt{n^3+1}}{\sqrt{n^4+1}+n} = \left[\frac{\infty}{\infty}\right] = \lim\limits_{n\to\infty}\frac{\sqrt{n^4\left(\frac{1}{n}+\frac{1}{n^4}\right)}}{\sqrt{n^4\left(1+\frac{1}{n^4}\right)}+n} = \lim\limits_{n\to\infty}\frac{\sqrt{n^4}\sqrt{\frac{1}{n}+\frac{1}{n^4}}}{\sqrt{n^4}\sqrt{1+\frac{1}{n^4}}+n} = \lim\frac{n^2\sqrt{\frac{1}{n}+\frac{1}{n^4}}}{n^2\sqrt{1+\frac{1}{n}}+n}$

dla $a,b>0$ $\sqrt{a\cdot b} = \sqrt{a}\cdot\sqrt{b}$

$\sqrt{a^4} = a^2$ $\quad \sqrt{a^2} = |a|$

$= \lim\limits_{n\to\infty}\frac{n^2\sqrt{\frac{1}{n}+\frac{1}{n^4}}}{n^2\left(\sqrt{1+\frac{1}{n}}+\frac{1}{n}\right)} = \left[\frac{0}{1+0}\right] = 0$

3) Oblicz granice ciągów:

a) $\lim\limits_{n\to\infty}\dfrac{1+2^n+4^{n+1}}{1+2^n+4\cdot 3^n}$

b) $\lim\limits_{n\to\infty}\dfrac{2^n-1}{2^{2n}-4}$

c) $\lim\limits_{n\to\infty}\dfrac{3^{n+1}-4^{n+2}}{3^{n+2}-2^{2n}-4}$

4) Oblicz granice ciągów:

Tw. 1. Jeśli $a_n \neq 0$ dla $n \in \mathbb{N}$ oraz $\lim\limits_{n\to+\infty} a_n = \pm\infty$, to $\lim\limits_{n\to+\infty}\left(1+\dfrac{1}{a_n}\right)^{a_n} = e$.

a) $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n^2}\right)^{2n^2}$

b) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{2n+1}\right)^{n-1}$

c) $\lim\limits_{n\to\infty}\left(1+\dfrac{2}{n^2+1}\right)^{2n-1}$

d) $\lim\limits_{n\to\infty}\left(1-\dfrac{1}{1-2n}\right)^{1-n^2}$

$\lim \frac{3n-1}{3n+3} = 1$

e) $\lim_{n\to\infty} \left(\frac{3n-1}{3n+3}\right)^n = [1^\infty] = \lim_{n\to\infty} \left(\frac{(3n+3)-4}{3n+3}\right)^n = \lim_{n\to\infty} \left(1 - \frac{4}{3n+3}\right)^n =$

$\lim_{n\to\infty} \frac{-4n}{3n+3} = -\frac{4}{3}$

$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{-\frac{3n+3}{4}}\right)^{-\frac{3n+3}{4}}\right]^{-\frac{4}{3n+3}\cdot n} = e^{-\frac{4}{3}}$ ($-\frac{3n+3}{4} \to -\infty$)

f) $\lim_{n\to\infty} \left(\frac{n^2-3}{n^2+2n+1}\right)^{-n^2} = [1^{-\infty}] = \lim_{n\to\infty} \left(\frac{(n^2+2n+1)-2n-4}{n^2+2n+1}\right)^{-n} =$

$\lim_{n\to\infty} \frac{(2n+4)n^2}{n^2+2n+1} = \infty$

$= \lim_{n\to\infty} \left(1 - \frac{2n+4}{n^2+2n+1}\right)^{-n} =$

$= \lim_{n\to\infty} \left[\left(1 + \frac{1}{-\frac{n^2+2n+1}{2n+4}}\right)^{-\frac{n^2+2n+1}{2n+4}}\right]^{\frac{2n+4}{n^2+2n+1}\cdot n^2} = [e^\infty] = +\infty$

5) Oblicz granice ciągów:

Tw. 2. $\lim_{n\to+\infty} \sqrt[n]{n} = 1.$

Tw. 3. Jeśli $a_n \geq 0$ dla $n \in \mathbb{N}$ oraz $\lim_{n\to+\infty} a_n = a > 0$, to $\lim_{n\to+\infty} \sqrt[n]{a_n} = 1.$

a) $\lim_{n\to\infty} \sqrt[n]{2 + 3^{-n} + 4^{1-2n}} \overset{TW3.}{=} 1$ bo $\lim (2 + 3^{-n} + 4^{1-2n}) = 2 > 0$

b) $\lim_{n\to\infty} \sqrt[n]{2 + 3^n + 4^{1+2n}} = [\infty^0] = \lim_{n\to\infty} \sqrt[n]{2 + 3^n + 4\cdot 16^n} = \lim \sqrt[n]{16^n \cdot \left(\frac{2}{16^n} + \frac{3^n}{16^n} + 4\right)}$

$= \lim \sqrt[n]{16^n} \cdot \sqrt[n]{\frac{2}{16^n} + \left(\frac{3}{16}\right)^n + 4} = 16 \cdot 1 = 16$

$\sqrt[n]{16^n} = 16$

$4 > 0$ (tw.3)

∞ nie możemy

c) $\lim_{n\to\infty} \sqrt[n]{n - 2n^2 + 4n^3} =$

$[\infty^0] = \lim_{n\to\infty} \sqrt[n]{n^3} \cdot \sqrt[n]{\frac{1}{n^2} - \frac{2}{n} + 4} =$

$= \lim_{n\to\infty} (\sqrt[n]{n})^3 \cdot \sqrt[n]{\frac{1}{n^2} - \frac{2}{n} + 4} = 1^3 \cdot 1 = 1$

$\sqrt[n]{n} \to 1$ (tw.2)

$4 > 0$, to stosujemy tw.3

d) $\lim_{n\to\infty} \sqrt[n]{\frac{4^n + 2^n}{4^{n+1} + 3^n + 1}} =$ c_n

1 sposób: $\lim_{n\to\infty} c_n = 1$ na mocy tw. 3, bo $\lim_{n\to\infty} \frac{4^n + 2^n}{4^{n+1} + 3^n + 1} = \frac{1}{4} > 0.$

2 sposób: $\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{\sqrt[n]{4^n + 2^n}}{\sqrt[n]{4^{n+1} + 3^n + 1}} = \{\text{dalej jak w b)}\} = \frac{4}{4} = 1$